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Application of homotopy perturbation method to the nonlinear pendulum

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ABSTRACT

The homotopy perturbation method is used to solve the nonlinear differential equation that governs the nonlinear oscillations of a simple pendulum, and an approximate expression for its period is obtained. Only one iteration leads to high accuracy of the solutions and the relative error for the approximate period is less than 2% for amplitudes as high as 130°. Another important point is that this method provides an analytical expression for the angular displacement as a function of time as the sum of an infinite number of harmonics, although for practical purposes it is sufficient to consider only a finite number of harmonics. We believe that the present study may be a suitable and fruitful exercise for teaching and better understanding perturbation techniques in advanced undergraduate courses on classical mechanics.

Keywords: Nonlinear pendulum; Approximate Period; Homotopy perturbation method.

1. Introduction

It is both instructive and interesting for students to analyze a familiar physical system in a new way and one of the most famous physical systems that allows us to apply different solution methods is the simple pendulum. An approximate linear model successfully describes many physical systems. However, nonlinear phenomena are encountered in all areas of sciences and engineering [1]. Perhaps one of the nonlinear systems most studied and analyzed is the simple pendulum, which is the most popular textbook example of a nonlinear system and is studied not only in advanced but also in introductory university courses of classical mechanics. The periodic motion exhibited by a simple pendulum is harmonic only for small angle oscillations [2]. Beyond this limit, the equation of motion is nonlinear. Small-angle approximation leads to an amplitude-independent period, while the large-amplitude solutions are not available in terms of elementary functions [3]. For pedagogical reasons one commonly starts the discussion of the motion of the simple pendulum by means of the simplest case of simple harmonic motion, valid only for small angle oscillations for which the approximation $\sin\theta \approx \theta$ can be made. In this approximation, the equation of motion is a linear differential equation and a relevant feature of such oscillatory motion is that the period doesn't depend on the amplitude. The linearized equation has a simple exact solution, whose derivation can be understood by first-year students [2-4]. However, the simple harmonic motion is unsatisfactory to model oscillation motion for large amplitudes and in such cases the period depends on the amplitude. Due to this, the discussion of oscillatory motion is important not only in most introductory courses on classical mechanics for undergraduate students but also in advances courses on this topic. It is important to show students that some problems can be solved exactly, but in most cases one has to be resorted to approximate solutions [5]. In this sense, it is true that the dynamics of a simple pendulum for small amplitudes is probably the example most widely used by educators to illustrate simple harmonic motion [6], but it is also true that the dynamics of a simple pendulum for large amplitudes is probably the most widely used example of a nonlinear oscillatory system.

Considerable attention has been directed towards the study of nonlinear problems in all areas of physics [7]. Nonlinear problems are intrinsic to different physical processes and for students it is very interesting to assimilate different techniques to solve these problems, since they are going to cope with them in the future. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one to a given nonlinear problem [8]. There are several methods used to find approximate solutions to nonlinear problems, such as perturbation techniques [9-12] or harmonic balance based methods [13-15]. A review of some asymptotic methods for strongly nonlinear equations can be found in detail in Ref. [16]. In general, given the nature of a nonlinear phenomenon, the approximate methods can only be applied within certain ranges of the physical parameters and/or to certain classes of problems [12].

Recently, we presented an application of the harmonic balance method to obtain an approximate expression for the period of the nonlinear pendulum [14]. The vital part of our argument was that an important advantage of this method is that it can be applied to nonlinear oscillatory problems for which the nonlinear terms are not "small", i.e., no perturbation parameter need exist [13]. For this type of oscillators –the nonlinear pendulum is one of them– the traditional perturbation methods such as the Lidstedt-Poincaré method cannot be applied because a linear term and a perturbation parameter are not present [10, 13, 17]. The use of a modified perturbation method is of great interest for students, since this technique can be applied to solve differential equations that couldn't be solved using standard perturbation methods. In our opinion students of physics must at least once analyse a nonlinear differential equation by using perturbation techniques, since these ones represent a relatively easy way of solving this type of equations without the use of complex mathematics. The modified perturbation method combines the flexibility and low complexity of the Lidstedt-Poincaré method with the advantage that it can be applied to differential equations where the Lidstedt-Poincaré method cannot be used.

In this paper we will obtain an approximate expression for the period of the nonlinear pendulum using a modified perturbation technique known as the homotopy perturbation method [16] proposed by J. H. He. In this method, which requires neither a small parameter nor a linear term in the differential equation, an artificial perturbation equation is constructed by embedding an artificial parameter $\varepsilon \in [0,1]$, which is used as an expanding parameter [13]. This technique yields a very rapid convergence of the solution series; in most cases only one iteration leads to high accuracy of the solution. This method provides an effective and convenient mathematical tool for nonlinear differential equations [13]. We think that this approach to perturbation theory can be used to convey to students the idea that perturbation techniques are powerful tools for approximately solving nonlinear problems. It is our opinion that the homotopy perturbation method applied here to the motion of a simple pendulum could be easily included in advanced classical mechanics courses for undergraduate students and even the calculation of the period could be presented in an introductory course of this subject.

2. Classical perturbation approaches

The differential equation modeling the free, undamped simple pendulum is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2 \sin\theta = 0 \tag{1}$$

where θ is the angular displacement, t is the time, $\omega_0^2 = g/l$ is the natural frequency of the small oscillations of the pendulum (when it is possible to do the approximation $\sin\theta \approx \theta$), l is the length of the pendulum and g is the acceleration due to gravity. The oscillations of the pendulum are subjected to the initial conditions $\theta(0) = A$ and $\theta'(0) = 0$, A being the amplitude of the oscillations. The periodic solution $\theta(t)$ of Eq. (1) and the period depend on the amplitude A. Eq. (1), although straightforward in appearance, is in fact difficult to solve because of the non-linearity of the trigonometric function.

The undergraduate introductory classical mechanics courses are usually restricted to the study of the oscillatory motion of the pendulum for small amplitudes. In this approximation we consider that the angle θ is small, and then the function $\sin\theta$ can be approximated by θ. Then the nonlinear differential equation (1) becomes a linear differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2\theta = 0 \tag{2}$$

which can be easily solved. Other textbooks and papers published in journals about physics education at university level present different analysis to solve the nonlinear equation (1). These may be based upon a perturbation treatment of the nonlinear equation of motion or a calculation of the period using elliptical integrals, and some texts and papers present both approaches [18]. When corrections to the approximation of simple harmonic motion using perturbation methods are obtained, aforementioned textbooks and papers begin their discussion by replacing the trigonometrical function $\sin\theta$ that appears in Eq. (1) by its power-series expansion which is truncated by considering different terms. If only the first two terms of the Taylor series expansion of $\sin\theta$ are kept, the simplest nonlinear approximation to Eq. (1) is obtained

$$\frac{d^2\theta}{dt^2} + \omega_0^2\theta + \varepsilon\theta^3 = 0, \qquad \varepsilon = -\frac{1}{6}\omega_0^2 \tag{3}$$

which corresponds to a Duffing oscillator [10] and it is a truncated version of the time dependence of the angular displacement of the pendulum. Eq. (3) has been approximately solved by means of different approximate techniques, and it is the paradigmatic nonlinear differential equation used to discuss and compare different perturbation approaches as can be seen, for example, in Marion's book [17] and both in early as recent educational papers published by Fulcher and Davis [18], Fernández [19], Amore et al [10] or Kahn and Zarmi [1]. The perturbation techniques usually considered in advanced courses of Classical Mechanics —and sometimes in introductory courses— are the Lindstedt-Poincaré and the alternative Lidstedt-Poincaré methods (see Refs. [17] and [10]), this is due to the fact than standard Lindstedt-Poincaré method is one of the most important techniques used in the study of nonlinear oscillations. In these approaches the solution of Eq. (3) and the unknown angular frequency are analytically expanded in the power series of the small parameter ε. When these perturbation techniques are explained to students, the initial point is to consider a second-order nonlinear differential equation similar to

$$\frac{d^2\theta}{dt^2} + \omega_0^2\theta + \varepsilon f(\theta) = 0 \tag{4}$$

The perturbation is written $\varepsilon f(\theta)$,where ε is a small parameter and when we set $\varepsilon = 0$ we obtain the linear differential equation (2) so we have a simple harmonic motion.

Obviously, that Eq. (3) has the same form of Eq. (4) and standard Lindstedt-Poincaré methods can be applied to Eq. (3) to obtain the period of motion and periodic approximate solutions for this equation. At this moment students understand that to apply standard Lindstedt-Poincaré methods they need a nonlinear differential equation with a linear term (in this case proportional to θ) and a small perturbation parameter (in this case ε). However, what can they do if a linear term or/and a perturbation parameter are not present in the differential equation? In fact, it is easy to show them some simple oscillatory systems for which the equation of motion is not as Eq. (4). One of them is presented in Marion's book [17] and corresponds to the motion of a mass attached to two identical stretched elastic wires for small amplitudes when the length of each wire without tension is the same as half the distance between the ends of the wires. This system does have for small amplitudes a linear term because the restoring force, $F(x)$, is proportional to the cube of the displacement, $F(x) = -\varepsilon x^3$. It is clear that none of the standard methods for constructing a perturbation solution can be applied to this system, since these procedures assume that when $\varepsilon = 0$ the resulting differential equation is that of the harmonic oscillator. It is evident, that the simple pendulum is another oscillatory system for which the equation of motion (1) is not as Eq. (4). If we compare Eq. (2) with Eq. (1), we conclude that Eq. (1) corresponds to a nonlinear oscillatory system for which there is no linear term and no perturbation parameter exists. In Eq. (1) the nonlinear function $\sin\theta$ cannot be treated as a perturbation and it is evident that the standard Lindstedt-Poincaré perturbation procedures cannot be applied to Eq. (1). However, we will show in the next section that with some "small" modifications it is possible to apply to Eq. (1) a perturbation technique similar to Lindstedt-Poincaré method, not being necessary to approximately solve this equation doing the power series expansion of the sinus function.

3. Approximate solution of the exact equation of the pendulum by homotopy perturbation method

By comparing Eq. (1) with Eq. (4), we can see that there is neither a linear term $\omega^2\theta$ nor a parameter ε in Eq. (1) and then standard perturbation methods cannot be applied, as we pointed out in the previous section. We also pointed out that one possibility is to solve Eq.

(1) approximately is to use the power-series expansion of the sine function and to consider only some terms of this expansion. Doing this it is easy to verify that an equation similar to Eq. (4) can be obtained and then standard perturbation methods such as the Lindstedt-Poincaré method could be applied. However, this implies finding an analytical approximate solution for an approximate nonlinear differential equation for the simple pendulum, but not a solution for the exact nonlinear differential equation. However, it is not very difficult and it is very illustrative to find an analytical approximate solution for the exact nonlinear Eq. (1). To do this, we are going to apply the homotopy perturbation method to solve Eq. (1). This method provides an approach to introducing an expanding parameter ε and a linear term $\omega^2\theta$ in the original Eq. (1).

The homotopy perturbation method is an effective method and can solve various nonlinear equations, in particular, nonlinear oscillatory systems. The idea of the homotopy perturbation method is very simple and straightforward. Consider the nonlinear differential Eq. (1) for the pendulum. This equation can be re-written in the form

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2\theta = \omega^2\theta - \omega_0^2 \sin\theta \tag{5}$$

where ω is the unknown angular frequency of the nonlinear pendulum. We then construct the so-called homotopy

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2\theta = \varepsilon(\omega^2\theta - \omega_0^2 \sin\theta) \tag{6}$$

where $\varepsilon \in [0,1]$ is an embedding parameter [8], ω is the unknown angular frequency of the oscillator and $\theta(t;\varepsilon)$ is a function of t and ε. When $\varepsilon = 0$ we have

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2\theta = 0 \tag{7}$$

which is a linear differential equation similar to Eq. (2). However in Eq. (7) the unknown angular frequency of the nonlinear pendulum, ω, appears instead of the angular frequency for the linear approximation, ω_0.

For $\varepsilon = 1$ we have

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2 \sin\theta = 0 \tag{8}$$

The solution to Eq. (7) is

$$\theta(t;0) \equiv \theta_0(t) = A\cos\omega t \tag{9}$$

and $\theta(t;1) \equiv \theta(t)$ is therefore obviously the solution to Eq. (8). As the embedding parameter ε increases from 0 to 1, the solution $\theta(t;\varepsilon)$ to Eq. (8) depends upon the embedding parameter ε and varies from the initial approximation $\theta_0(t)$ to the solution $\theta(t)$ of Eq. (8). In topology, such a kind of continuous variation is called "deformation" [8, 16]. Based on the idea of homotopy, a modified Lindstedt-Poincaré perturbation method was developed by He [16]. The idea is to apply the so-called alternative Lidstedt-Poincaré [10, 17] method to Eq. (6) where the embedding parameter ε is considered a "small" perturbation parameter. Therefore, the method is called the homotopy perturbation method, which combines the advantages of the standard perturbation methods and those of the homotopy techniques. In this method the perturbation Eq. (6) can be easily constructed by homotopy and the initial approximation (9) can also be freely selected. The basic assumption of this perturbation technique is that the solution to Eq. (6) can be expressed as a series of the homotopy (perturbation) parameter ε

$$\theta(t) = \theta_0(t) + \varepsilon\theta_1(t) + \varepsilon^2\theta_2(t) + \ldots \tag{10}$$

The convergence of this series is discussed in Ref. [20], while the asymptotic character of Eq. (10) is shown in Refs. [16, 21].

Substituting Eq. (10) into Eq. (6), and equating the terms with identical powers of ε, we can obtain a series of linear equations, of which we write only the first two

$$\frac{\mathrm{d}^2\theta_0}{\mathrm{d}t^2} + \omega^2\theta_0 = 0, \qquad \theta_0(0) = A, \qquad \frac{\mathrm{d}\theta_0(0)}{\mathrm{d}t} = 0 \tag{11}$$

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} + \omega^2\theta_1 = \omega^2\theta_0 - \omega_0^2 \sin\theta_0, \qquad \theta_1(0) = 0, \qquad \frac{\mathrm{d}\theta_1(0)}{\mathrm{d}t} = 0 \tag{12}$$

The solution to Eq. (11) is

$$\theta_0(t) = A\cos\omega t \tag{13}$$

which describes the simple oscillatory motion of the pendulum with an unknown angular frequency ω. Substituting Eq. (13) into Eq. (12) gives

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} + \omega^2\theta_1 = \omega^2 A\cos\omega t - \omega_0^2 \sin(A\cos\omega t) \tag{14}$$

For there to be no secular terms in θ_1, contributions proportional to $\cos\omega t$ on the right hand side of Eq. (14) must be eliminated. It is necessary to eliminate these terms because when we obtain the solution to the linear differential Eq. (14), a term proportional to $\cos\omega t$ introduces a term proportional to $t\sin\omega t$ in the solution $\theta_1(t)$. This term is not only obviously nonperiodic but also approaches infinity when t increases. The so-called secular terms are terms such as $t^m\cos\omega t$ or $t^m\sin\omega t$. These terms arise because the expansion of Eq. (10), where $\theta(t)$ is a periodic solution of the nonlinear differential equation, is non-uniformly valid. The existence of such terms destroys the periodicity of the expansion given by Eq. (10) when only a finite number of terms is used. These terms make $\theta_1(t)$, the correction term in the periodic solution $\theta_0(t)$, not only non-periodic, but, in addition, unbounded as t approaches infinity. Therefore, to construct a uniformly valid solution, an approximation is needed that eliminates secular terms. A technique to avoid the presence of secular terms was developed by Lindstedt, and later Poincaré proved that the expansions obtained by Lindstedt's method are both asymptotic and uniformly valid. As we can see, introduction of the new independent variable ω allows the secular term to be eliminated [13].

To obtain and eliminate secular terms in Eq. (14) we need to obtain contributions proportional to $\cos\omega t$ due to $\sin(A\cos\omega t)$. To do this, we can consider the following Taylor series expansion

$$\sin\theta_0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,\theta_0^{2n+1} \tag{15}$$

Substituting Eq. (13) into Eq. (15), we obtain

$$\sin(A\cos\omega t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,A^{2n+1}\cos^{2n+1}\omega t \tag{16}$$

The formula that allows us to obtain the odd power series of the cosine is

$$\cos^{2n+1}\omega t = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{n-k} \cos[(2k+1)\omega t] \tag{17}$$

Substituting Eq. (17) into Eq. (16) gives

$$\sin(A\cos\omega t) = \sum_{n=0}^{\infty} \frac{(-1)^n A^{2n+1}}{2^{2n}(2n+1)!} \sum_{k=0}^{n} \binom{2n+1}{n-k} \cos[(2k+1)\omega t] \tag{18}$$

which can be written as follows

$$\sin(A\cos\omega t) = \sum_{j=0}^{\infty} a_{2j+1}\cos[(2j+1)\omega t] \tag{19}$$

where

$$a_{2j+1} = \sum_{n=j}^{\infty} \frac{(-1)^n A^{2n+1}}{2^{2n}(2n+1)!}\binom{2n+1}{n-j} = \sum_{n=j}^{\infty} \frac{(-1)^n A^{2n+1}}{2^{2n}(n-j)!(n+j+1)!} = 2(-1)^j J_{2j+1}(A) \tag{20}$$

In Eq. (20), J_{2j+1} is the $(2j+1)$-order Bessel function of the first kind which, taking into account that $j = 0, 1, 2, \ldots$, is defined as follows

$$J_{2j+1}(A) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2j+2k+1}k!(2j+k+1)!}A^{2j+2k+1} = \frac{1}{2}(-1)^j \sum_{n=j}^{\infty} \frac{(-1)^n}{2^{2n}(n-j)!(n+j+1)!}A^{2n+1} \tag{21}$$

and the simplifications have been done using MATHEMATICA.

Eq. (19) can be written as follows

$$\sin(A\cos\omega t) = a_1\cos\omega t + \sum_{j=1}^{\infty} a_{2j+1}\cos[(2j+1)\omega t] \tag{22}$$

and substituting the above equation into Eq. (14) gives

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} + \omega^2\theta_1 = \omega^2 A\cos\omega t - \omega_0^2 a_1 A\cos\omega t - \omega_0^2 \sum_{j=1}^{\infty} a_{2j+1}\cos[(2j+1)\omega t] \tag{23}$$

As we pointed out above, removing secular terms in θ_1 is necessary to set the coefficient of $\cos\omega t$ equal to zero in Eq. (23). Consequently, the unknown angular frequency ω can be determined as follows

$$\omega(A) = \omega_0 \sqrt{\frac{a_1}{A}} = \omega_0 \sqrt{\frac{2J_1(A)}{A}} \tag{24}$$

and the approximate period T_{app} of oscillations can be obtained as follows

$$\frac{T_{app}(A)}{T_0} = \sqrt{\frac{A}{2J_1(A)}} \tag{25}$$

where $T_{app} = 2\pi/\omega$, ω is given in Eq. (24), and $T_0 = 2\pi/\omega_0$ is the period for the small amplitudes. As we can see, at this approximation the present method gives exactly the same result for the approximate period as does the first order harmonic balance method [14].

To illustrate the remarkable accuracy of the result obtained, we compare the approximate period with the exact one. The exact period T_{ex} of the nonlinear pendulum is expressed in the form [6]

$$\frac{T_{ex}}{T_0} = \frac{2}{\pi} K(q) \tag{26}$$

where the modulus q is equal to $\sin(A/2)$ and $K(q)$ is the complete elliptical integral of the first kind given by the following equation

$$K(q) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - q^2 \sin\varphi}} \tag{27}$$

In order to illustrate the accuracy of the approximate period T_{app}, we compare Eqs. (25) and (27). Firstly we will compare the power series expansions on the right hand side of these equations. Using this series in the integrand of Eq. (27) for $q < 1$, reduces the elliptical integral there to a sum of elementary integrals [17]. Retaining only the first four terms, we have that

$$\frac{T_{ex}(q)}{T_0} = 1 + \frac{1}{4}q^2 + \frac{27}{192}q^4 + \frac{1125}{11520}q^6 + \ldots \tag{28}$$

Taking into account that $q = \sin(A/2)$, and with the help of the MATHEMATICA program, it is possible to do the following series expansion of the approximate period (Eq. (25))

$$\frac{T_{app}(q)}{T_0} = \sqrt{\frac{\arcsin(q)}{J_1(2\arcsin(q))}} = 1 + \frac{1}{4}q^2 + \frac{26}{192}q^4 + \frac{1042}{11520}q^6 + \ldots \tag{29}$$

As we can see, the first two terms in Eq. (29) are the same as the first two terms of Eq. (28) obtained in the power-series expansion of the exact period, whereas the third term of the expansion of the exact period is 27/192 compared with 26/192 obtained in our study. In other words, the relative error in this term is 3.7%. In the fourth term a coefficient 1042/11520 is obtained, whereas the exact value is 1125/11520, that is, the relative error in this term is 7.4%. As we can see, the relative errors of these two terms are not high.

Now we consider the percentage of relative error defined by the equation

$$\Delta(A) = 100\left|\frac{T_{app}(A) - T_{ex}(A)}{T_{ex}(A)}\right| \tag{30}$$

In Figure 1 we plotted the percentage of relative error, $\Delta(A)$, as a function of the amplitude of oscillations, A. As can be seen in this figure, the relative error of the approximate period is less than 1%, 2% and 5% for $A < 116°$, $A < 131°$ and $A < 150°$, respectively.

Now in order to obtain the correction term θ_1 for the periodic solution θ_0 we are going to take into account the following procedure. Taking into account Eqs. (19) and (23), we can now write equation (14) as follows

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} + \omega^2\theta_1 = -\omega_0^2\sum_{j=1}^{\infty} a_{2j+1}\cos[(2j+1)\omega t] \tag{31}$$

with initial conditions

$$\theta_1(0) = 0\,, \qquad \frac{\mathrm{d}\theta_1(0)}{\mathrm{d}t} = 0 \tag{32}$$

Eq. (31) is a linear second-order inhomogeneous differential equation. The solution to Eq. (31) can be written as follows

$$\theta_1(t) = \theta_{1h}(t) + \theta_{1p}(t) \tag{33}$$

where $\theta_{1h}(t)$ is the solution to the homogeneous differential equation

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} + \omega^2\theta_1 = 0 \tag{34}$$

and $\theta_{1p}(t)$ is called the particular solution to Eq. (33). The homogeneous solution contains two arbitrary constants, while the particular solution does not contain any arbitrary constants. The solution to Eq. (32) can be written as follows

$$\theta_1(t) = Ac_1 \cos\omega t + A\sum_{j=1}^{\infty} c_{2j+1} \cos[(2j+1)\omega t] = A\sum_{j=0}^{\infty} c_{2j+1} \cos[(2j+1)\omega t] \tag{35}$$

where

$$\theta_{1h}(t) = Ac_1 \cos\omega t \qquad \text{and} \qquad \theta_{1p}(t) = A\sum_{j=1}^{\infty} c_{2j+1} \cos[(2j+1)\omega t] \tag{36}$$

and Eq. (32) has been taken into account. Substituting Eq. (35) into Eq. (31) gives

$$-A\omega^2\sum_{j=0}^{\infty}(2j+1)^2 c_{2j+1}\cos[(2j+1)\omega t]+A\omega^2\sum_{j=0}^{\infty}c_{2j+1}\cos[(2j+1)\omega t]=-\omega_0^2\sum_{j=1}^{\infty}a_{2j+1}\cos[(2j+1)\omega t]$$

(37)

From the above equation, and taking into account that $-(2j+1)^2+1=-4j(j+1)$, we obtain the following expression for the coefficients c_{2j+1} (for $j > 1$)

$$c_{2j+1}=\frac{a_{2j+1}}{4j(j+1)A}\frac{\omega_0^2}{\omega^2}=\frac{a_{2j+1}}{4j(j+1)a_1}=\frac{(-1)^j J_{2j+1}(A)}{4j(j+1)J_1(A)} \tag{38}$$

where Eqs. (20) and (24) have been taken into account. To obtain the value of the coefficient c_1 we consider that $\theta_1(0)=0$, then Eq. (35) gives

$$\theta_1(0)=\sum_{j=0}^{\infty}c_{2j+1}=0 \tag{39}$$

and the value of coefficient c_1 is given by the following expression

$$c_1=-\sum_{j=1}^{\infty}c_{2j+1}=\frac{1}{4J_1(A)}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}J_{2j+1}(A)}{j(j+1)} \tag{40}$$

Finally, with this approximation we obtain the following analytical solution for Eq. (1),

$$\theta(t;A)\equiv\theta(t)=A\left(1-\sum_{j=1}^{\infty}c_{2j+1}(A)\right)\cos[\omega(A)t]+A\sum_{j=1}^{\infty}c_{2j+1}(A)\cos[(2j+1)\omega(A)t] \tag{41}$$

where $\omega(A)$ is given by Eq. (24) and we have taken into account in Eq. (10) that $\varepsilon = 1$. As we can see, $\theta(t)$ has an infinite number of harmonics. However, it is possible to truncate the series expansion at Eq. (41) and to write an approximate equation, $\theta^{(N)}(t)$, for the "approximate solution" $\theta(t)$ as follows:

$$\theta^{(N)}(t) = A\left(1-\sum_{j=1}^{N} c_{2j+1}(A)\right)\cos[\omega(A)t] + A\sum_{j=1}^{N} c_{2j+1}(A)\cos[(2j+1)\omega(A)t] \quad (42)$$

which has only a finite number of harmonics. Comparing Eqs. (41) and (42), it follows that

$$\lim_{N\to\infty} \theta^{(N)}(t) = \theta(t) \quad (43)$$

It is possible to do this because the values of c_{2j+1} are small when j is increased, and then it is possible to consider only a few terms in Eqs. (40) and (41). For example, from Eq. (38) and for A = 116° we obtain c_3 = -0.029, c_5 = 0.00054, …, and for amplitudes A < 116° it would be possible to consider only up to j = 1 and Eq. (41) could be approximated as follows

$$\begin{aligned}\theta(t) \approx \theta^{(1)}(t) &= [1-c_3(A)]A\cos(\omega(A)t) + c_3 A\cos(3\omega(A)t) \\ &= \left(1+\frac{J_3(A)}{8J_1(A)}\right)A\cos(\omega(A)t) - \frac{J_3(A)}{8J_1(A)}A\cos(3\omega(A)t)\end{aligned} \quad (44)$$

Also, for amplitudes A < 116° it would be possible to use the following expression for the angular displacement of the pendulum $\theta(t) \approx A\cos\omega t$, which coincides with the value obtained by applying the first order harmonic balance method [14].

4. Further discussion

Finally, the solution procedure considered here is only valid for first-order approximate solutions. However, to eliminate the secular terms in higher order approximations, it is easy to construct another homotopy and apply the alternative Lindstedt-Poincaré method summarized by Amore et al in this journal [10]. To do this it is necessary to write the dimensionless form of the original nonlinear differential equation

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + f(x(t)) = 0 \tag{45}$$

and construct a homotopy in the following form

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + x(t) = \varepsilon\left[x(t) - f(x(t))\right] \tag{46}$$

The alternative Lindstedt-Poincaré method provides a ε-power series for the square of the unknown frequency ω of the motion as follows [10, 17]

$$\omega^2 = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots \tag{47}$$

The homotopy perturbation approach to remove secular terms for higher orders is based on the substitution of this expansion into the Eq. (46)

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + (\omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \ldots)x(t) = \varepsilon\left[x(t) - f(x(t))\right] \tag{48}$$

Then the expansion of $x(t)$ in powers of ε is substituted in this equation and terms with identical powers of ε are equated, giving a set of linear differential equations like Eqs. (11) and (12).

5. Conclusions

To conclude, we have applied the homotopy perturbation method to obtain an approximate expression for the period of the nonlinear pendulum. This provides a simple and pedagogical example for introducing this alternative perturbation technique by means of an example for which the standard perturbation techniques such as the standard Lidstedt-Poincaré method cannot be applied. Undergraduate classical mechanics textbooks usually do not provide information about this technique. They usually solve the equation for the nonlinear pendulum expanding $\sin\theta$ in power series, and apply the classical perturbation methods to an approximated nonlinear differential equation such as Eq. (3). Then, an approximate solution is found for an approximate nonlinear differential equation (nor for the exact equation). The usual Lindstedt-Poincaré perturbation method in not applicable for solving Eq. (1) because there is no linear term and no perturbation parameter exists. However, we have shown in this paper that with some “small” modifications it is possible to apply the alternative Lindstedt-Poincaré method, a technique that is usually explained in courses of classical mechanics. On the other hand, undergraduate students can easily see that the homotopy perturbation method considered in this paper is extremely simple in its principle, quite easy to use, and gives a very good numerical accuracy. Therefore, it could be easily included in lectures on classical mechanics for undergraduate students. Moreover, we point out that in the process of solving the perturbation equations and in other calculations one may show the students the advantage of using available symbolic computer programs such as MATHEMATICA.

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FIGURE CAPTION

Figure 1.- Percentage of relative error, $\Delta(A)$ (equation (31)), as a function of the amplitude of oscillations, A.

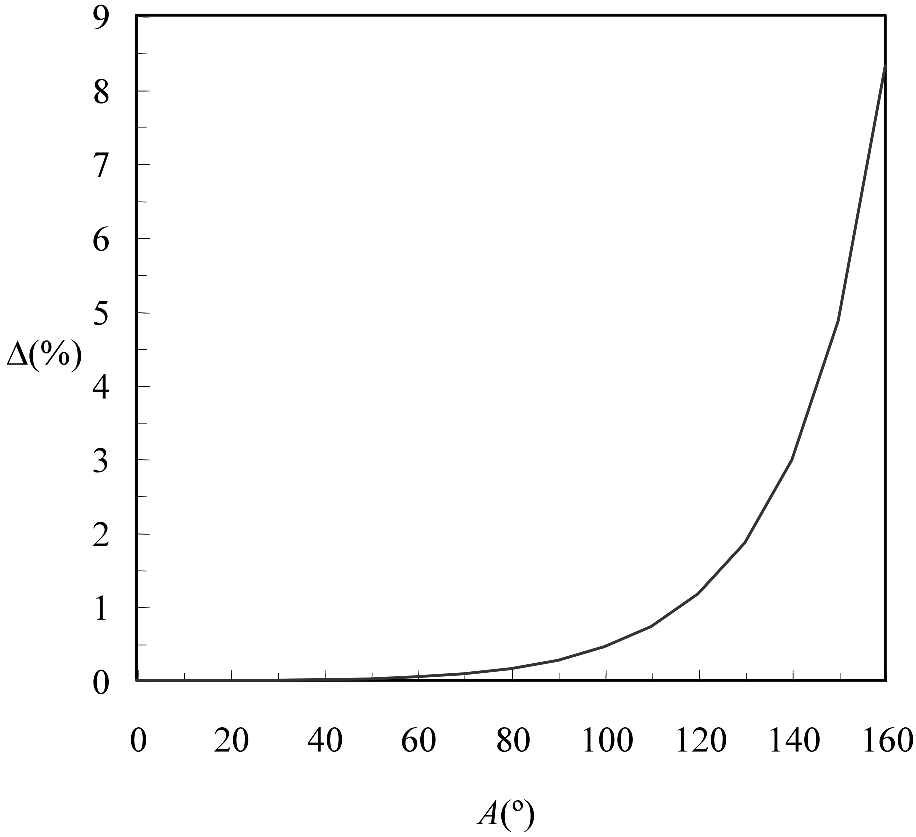

FIGURE 1